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AN ELEMENTARY PROOF OF THE HEWITT-SHIROTA THEOREM

by

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ZW 1968-011

may 1968

The Mathematical Centre at Amsterdam, founded the 11th of February, 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications, and is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) and the Central Organization for Applied Scientific Research in the Netherlands (T.N.O.), by the Municipality of Amsterdam and by several industries.

The abbreviation "space" is used to denote "completely regular space".

A well-known theorem of Hewitt and Shirota [1] states that a realcompact completely regular space is homeomorphic with a closed subspace of a product of real lines. Many proofs of this fundamental theorem have appeared, among which are those applying general realcompactification methods and methods using embedding in complete uniform structures. The present note has the goal of giving an elementary self-contained proof of this theorem invoking only basic set theoretical properties of realcompactness.

Recall that a space is realcompact provided that each maximal centered family of zerosets with the countable intersection property has non empty intersection.

If X is a space, then it is well-known (see [1] page 17) that each two disjoint zerosets of X have disjoint cozeroset neighbourhoods. Moreover, each finite cover of X by cozerosets has a finite refinement consisting of zerosets. The proof of the last remark is similar to the proof of the statement that each finite open cover of a normal space has a finite closed refinement.

If X is a space and αX an extension of X in which X is dense, then we shall say that αX has property (Z) in case that for each countable collection of zerosets of X with empty intersection, the closures in αX have empty intersection.

LEMMA 1. Let αX be an extension of X which contains X as a dense subspace and such that each continuous function on X has a continuous extension over αX. Then αX has property (Z).

PROOF. If Z_1 and Z_2 are disjoint zerosets of X, then by the complete regularity of X there exists $f \in C(X)$ satisfying $f(Z_1) = 0$, $f(Z_2) = 1$. Let $\bar{f}$ be the continuous extension of f over αX. It follows that $\bar{Z}_1^{\alpha X} \cap \bar{Z}_2^{\alpha X} \subset \bar{f}^{-1}(0) \cap \bar{f}^{-1}(1) = \emptyset$. Moreover, if $\{Z_1,\ldots,Z_n\}$ is a finite collection of zerosets of X with empty intersection, then according to the remark above there exists a finite collection of zerosets $\{T_1,\ldots,T_m\}$ which is a cover of X and which refines $\{X\setminus Z_1,\ldots,X\setminus Z_n\}$.

The fact that each two disjoint zerosets of X have disjoint closures in αX implies that $\{\bar{T}_i^{\alpha X} | i=1,2,..,m\}$ is a cover of αX which refines $\{\alpha X \setminus \bar{Z}_i^{\alpha X} | i=1,2,..,n\}$. Hence $\bigcap \{\bar{Z}_i^{\alpha X} | i=1,2,...,n\} = \emptyset$.

Now, let $\{Z_i | i=1,2,...\}$ be a countable collection of zerosets of X with empty intersection. If there exists $p \in \bigcap \{\bar{Z}_i^{\alpha X} | i=1,2,...\}$, then for i=1,2,... let $f_i \in C(X)$ be such that $0 \leq f_i \leq 1$ and $Z_i = \{x \in X | f_i(x) = 0\}$.
The result proved in the last few lines above implies that an arbitrary (zeroset) neighbourhood U of p in αX intersects $Z_1 \cap Z_2 \cap \ldots \cap Z_k$ for each k, so the function f on X defined by

$$f(x) = \sum_{i=1}^{\infty} 2^{-i} f_i(x)$$

takes arbitrarily small values on $U \cap X$. It follows that the function $1/f$ cannot be extended continuously over αX, which contradicts our hypothesis.

LEMMA 2. <u>If X is a realcompact space and if αX is an extension of X with propery (Z), then $\alpha X = X$.</u>

PROOF. Denote the collection of zerosets of X by $\mathfrak{Z}$. Assume that there exists $p \in \alpha X \setminus X$, and let $\mathfrak{Z}_1$ be the subcollection of $\mathfrak{Z}$ defined by $\mathfrak{Z}_1 = \{Z \in \mathfrak{Z} | p \in \bar{Z}^{\alpha X}\}$. Condition (Z) implies that $\mathfrak{Z}_1$ is a maximal centered family of zerosets of X with the countable intersection property; thus by realcompactness of X there exists $q \in \bigcap \mathfrak{Z}_1$. Let G be a zeroset neighbourhood of p in αX which contains p and does not meet q. Then $p \in \overline{G \cap X}^{\alpha X}$, so $G \cap X$ is a member of $\mathfrak{Z}_1$ which does not meet q. This is a contradiction.

We are now in a position to prove Hewitt-Shirota's theorem. We state it in the following way.
THEOREM. <u>Let X be a realcompact space. The mapping $e: X \to \mathbb{R}^{C(X)}$ defined by $e(x)_f = f(x)$ for $f \in C(X)$ is a homeomorphism of X onto a closed subspace of $\mathbb{R}^{C(X)}$.</u>

PROOF. By the complete regularity of X, e is a homeomorphism. By lemma 1 the closure $\overline{e(X)}$ of $e(X)$ in $\mathbb{R}^{C(X)}$ is an extension of X with property (Z) and by lemma 2, $\overline{e(X)} = e(X)$. Thus e is a closed embedding.

REFERENCES. [1] L. Gillman and M. Jerison, Rings of continuous functions, van Nostrand, 1960.

[2] P. Alexandroff and H. Hopf, Topologie, Berlin, 1935.